AN ANALYSIS OF INTEGRATED OBSERVATIONS

BY T. V. AVADHANI

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1. INTRODUCTION

While explaining the behaviour of the observed correlograms of Mercer and Hall [4] wheat data, Whittle [6, p. 443] remarked in his paper as follows :

> "...we must recall that the observations are not *point* observations of growth but *integrated* observations of the growth over an area. An integration such as this will enhance the auto-covariance of zero lag relative to the others..."

Bartlett [2, p. 265] remarked that :

> "...it is some times more reasonable to envisage a process of continuous type from which discrete observations are made..."

Whittle [6] and Patankar [5] analysed the wheat data by taking the observations representing the growth over small rectangular plots over which the yields are actually observed, as point observations of growth at the centers of these rectangles coming from a discrete two dimensional Stationary Stochastic process, assuming a theoretical model for the auto-correlation function.

Here two important considerations arise when there are observations at a discrete set of points :

(*i*) Are the data to be treated as coming from a two dimensional discrete process or as observations at discrete points of a two dimensional continuous process ?

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(*ii*) Are the observations to be treated as values at a point or are they to be treated as representing the values of growth over a small rectangular plot about this point as centre, thereby constituting observations over a neighbourhood of it ? In the latter case, which is what actually obtains in practice, how is this effect of integration to be accounted for ?

In this paper an attempt is made in section 3 to obtain the auto-correlation of the 'Integrated process' as explicitly specified in section 2, in terms of the auto-correlation of the 'Basic process' which is taken as a two dimensional Stationary Stochastic process. Assuming that the auto-correlation $\rho(s, t)$ of the 'Basic process' is $\rho_1^{|s|} \rho_2^{|t|}$, corresponding to a two dimensional linear Markoff Process an estimate and the standard error of the estimate of the sample auto-correlation of lag (s, t) of the Integrated process are obtained using the results from [1].

In section 4, the results of section 3 are applied to Mercer and Hall wheat data to obtain estimates and standard errors of the estimates of the sample auto-correlations of lags (1,0) of the Integrated process assuming the theoretical model for $\rho(s, t)$ as $\rho_1^{|s|} \rho_2^{|t|}$, which is the one employed by Whittle and Patankar. The result obtained in this paper which take into account the effect of integration are compared with the results obtained by the two authors who did not take into account the effect of integration.

2. PRELIMINARIES

2·1. Notation :

Let $\xi(x, y)$ be a two dimensional Stationary Stochastic process with mean zero, of the continuous parameters x and y. Let the process be continuous in the mean so that the integral of the process has a meaning in the sense of Cramer. This process is hereafter referred to as the 'Basic process'.

Let

$$Z(x, y) = \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} \int_{y-\frac{k}{2}}^{y+\frac{k}{2}} \xi(u, v)\, du\, dv \quad (h, k > 0)$$

It is clear that $Z(x, y)$ is also a two dimensional Stationary Stochastic continuous parameter process. This process is hereafter referred to as the 'Integrated process' of the 'Basic process' $\xi(x, y)$. The theoretical and observed auto-correlations of lag (s, t) of the Basic process and the associated Integrated process are denoted by $\rho(s, t)$, $r(s, t)$, $\rho_1(s, t)$ and $R(s, t)$ respectively. (The formulae used for the auto-correlations are those in [6]).

2·2. Lemmas :

Lemma 1. For $|\rho| < 1$ and $h > 0$

$$\int_{s-\frac{h}{2}}^{s+\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^{|u-v|}\, du\, dv$$

$$= \begin{cases} \dfrac{-2(h-|s|)}{\log|\rho|} + \dfrac{\rho^{|s|}(\rho^{h}-2)+\rho^{h-|s|}}{(\log|\rho|)^2} & \text{if } |s| < h \\[2ex] \dfrac{\rho^{|s|}(\rho^{-h/2}-\rho^{h/2})^2}{(\log|\rho|)^2} & \text{if } |s| \geqslant h \end{cases} \qquad \ldots(2{\cdot}2{\cdot}1)$$

Proof:

(*i*) Let $|s| < h$. If $0 \leqslant s < h$

$$\int_{s-\frac{h}{2}}^{s+\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^{|u-v|}\, du\, dv = \int_{s-\frac{h}{2}}^{s+\frac{h}{2}} \int_{-\frac{h}{2}}^{s-\frac{h}{2}} \rho^{u-v}$$

$$du\, dv + \int_{s-\frac{h}{2}}^{h/2} \left(\int_{v}^{s+\frac{h}{2}} \rho^{u-v}\, du \right) dv + \int_{s-\frac{h}{2}}^{h/2} \left(\int_{s-\frac{h}{2}}^{v} \rho^{v-u} du \right) dv$$

which on integration gives

$$\frac{-2(h-s)}{\log|\rho|} + \frac{\rho^{s}(\rho^{h}-2)\rho+^{h-s}}{(\log|\rho|)^2} \qquad \ldots(2{\cdot}2{\cdot}2)$$

If $-h < s \leqslant 0$ it is easily observed that a similar result is obtained by replacing s in (2·2·2) by $-s$. Combining these two results we obtain (2·2·1) for $|s| < h$.

(*ii*) Let $|s| \geqslant h$. If $s \geqslant h$.

$$\int_{s-\frac{h}{2}}^{s+\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^{|u-v|}\, du\, dv = \int_{s-\frac{h}{2}}^{s+\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^{u-v}\, du\, dv$$

which on integration gives

$$\frac{\rho^s}{(\log|\rho|)^2}(\rho^{-h/2}-\rho^{h/2})^2 \qquad \ldots(2\cdot2\cdot3)$$

If $s \leqslant -h$, it is seen that a result similar to (2·2·3) is obtained by replacing s by $-s$. Combining these two results we obtain (2·2·1) for $|s| \geqslant h$.

Lemma 2.

If $$\rho_1(u)=\begin{cases}\lambda\rho^{|u|} & \text{if } |u|\geqslant 1\\ 1 & \text{if } u=0\end{cases} \qquad \ldots(2\cdot2\cdot4)$$

where $|\rho|<1$, then

(*i*) $$\sum_{u=-\infty}^{\infty}\rho_1^2(u)=1+\frac{2\lambda^2\rho^2}{1-\rho^2} \qquad \ldots(2\cdot2\cdot5)$$

and

(*ii*) $$\sum_{u=-\infty}^{\infty}\rho_1(u-s)\,\rho_1(u+s')=\rho^{s+s'}\left\{2\lambda+\lambda^2\left(s+s'-2+\frac{1+\rho^2}{1-\rho^2}\right)\right\} \qquad \ldots(2\cdot2\cdot6)$$

for s and $s' \neq 0$.

Proof:

(*i*) Using (2·2·4) and writing

$$\sum_{u=-\infty}^{\infty}\rho_1^2(u)=1+2\lambda^2\sum_{u=1}^{\infty}\rho^{2u}$$

it is easily seen that (2·2·5) follows.

(*ii*) Using (2·2·4), the left hand side of (2·2·6) can be written as

$$\sum_{u=-\infty}^{-s'-1}\lambda^2\rho^{|u-s|+|u+s'|}+\sum_{u=-s'+1}^{s-1}\lambda^2\rho^{|u-s|+|u+s'|}$$

$$+\sum_{u=s+1}^{\infty}\lambda^2\rho^{|u-s|+|u+s'|}+2\lambda\rho^{s+s'} \qquad \ldots(2\cdot2\cdot7)$$

By adding and subtracting $2\lambda^2\rho^{s+s'}$ to (2·2·7) it becomes

$$\sum_{u=-\infty}^{\infty}\lambda^2\rho^{|u-s|+|u+s'|}+2\lambda\rho^{s+s'}-2\lambda^2\rho^{s+s'}$$

which by [1, (5·1)] is

$$=\lambda^2\rho^{s+s'}\left(s+s'+\frac{1+\rho^2}{1-\rho^2}\right)+2\lambda\rho^{s+s'}-2\lambda^2\rho^{s+s'}$$

from which (2·2·6) follows.

3. Main Results

In this section $\rho_I(s, t)$ is expressed in terms of $\rho(s, t)$. As $R(s, t)$ is an unbiased and consistent estimate of $\rho_I(s, t)$, a consistent estimate $r(s, t)$ of $\rho(s, t)$ is found in terms of $R(1, 0)$ and $R(0, 1)$ assuming a theoretical model for $\rho(s, t)$ as $\rho_1^{|s|}\rho_2^{|t|}$. The standard error of $R(s, t)$ in general and the errors of $R(1, 0)$ and $R(0, 1)$ in particular are also worked out.

If $R(s, t)$ is a consistent estimate of $\rho_I(s, t)$, then $r(s, t)$ would be a consistent estimate of $\rho(s, t)$ and the asymptotic variance of $r(s, t)$ can be worked out from that of $R(s, t)$. Any asymptotic efficiency properties of $R(s, t)$ are carried over to $r(s, t)$.

3·1. Result on $\rho_I(s, t)$:

Theorem 1: If $\rho_I(s, t)$ and $\rho(s, t)$ are the auto-correlations of lag (s, t) of the Integrated process and the Basic process which is Stationary, then

$$\rho_I(s, t)=\frac{\displaystyle\int_{s-\frac{h}{2}}^{s+\frac{h}{2}}\int_{t-\frac{k}{2}}^{t+\frac{k}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{k}{2}}^{\frac{k}{2}}\rho(x-u, y-v)\,dxdydudv}{\displaystyle\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{k}{2}}^{\frac{k}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{k}{2}}^{\frac{k}{2}}\rho(x-u, y-v)\,dxdydudv} \tag{3·1·1}$$

Proof:

$$\sigma^2_Z\,\rho_I(s, t)=E\{Z(a+s, b+t)\,Z(a, b)\}$$

$$=\int_{a+s-\frac{h}{2}}^{a+s+\frac{h}{2}}\int_{b+t-\frac{k}{2}}^{b+t+\frac{k}{2}}\int_{a-\frac{h}{2}}^{a+\frac{h}{2}}\int_{b-\frac{k}{2}}^{b+\frac{k}{2}}E\{\xi(x, y)\,\xi(u, v)\}dxdy\,dudv$$

$$=\sigma^2_\xi\int_{a+s-\frac{h}{2}}^{a+s+\frac{h}{2}}\int_{b+t-\frac{k}{2}}^{b+t+\frac{k}{2}}\int_{a-\frac{h}{2}}^{a+\frac{h}{2}}\int_{b-\frac{k}{2}}^{b+\frac{k}{2}}\rho(x-u, y-v)\,dxdydudv \quad \ldots(3·1·2)$$

By putting $s=0$ and $t=0$ in the integral on the right side (3·1·2) σ^2_Z is obtained. Substituting this for σ^2_Z on the left hand side of (3 1·2) and shifting the variables to $a+s$, $b+t$, a, and b it is easily seen that (3·1·1) follows.

THEOREM 2. If the auto-correlation of lag (s, t) of the Basic process is $\rho_1^{|s|}\rho_2^{|t|}$, then the auto-correlation of the Integrated process $\rho^I(s, t)$ is

$$\frac{\left\{\frac{\rho_1^{|s|}(\rho_1^h-2)+\rho_1^{h-|s|}}{(\log|\rho_1|)^2}-\frac{2(h-|s|)}{\log|\rho_1|}\right\}\left\{\frac{\rho_2^{|t|}(\rho_2^k-2)+\rho_2^{k-|t|}}{(\log|\rho_2|)^2}-\frac{2(k-|t|)}{\log|\rho_2|}\right\}}{4\left\{\frac{\rho_1^h-1}{(\log|\rho_1|)^2}-\frac{h}{\log|\rho_1|}\right\}\left\{\frac{\rho_2^k-1}{(\log|\rho_2|)^2}-\frac{k}{\log|\rho_2|}\right\}}$$

for $|s|<h$ and $|t|<k$, (3·1·3)

$$\rho_1^{|s|}\rho_2^{|t|}\frac{(\rho_1^{-\frac{h}{2}}-\rho_1^{\frac{h}{2}})^2(\rho_2^{-\frac{k}{2}}-\rho_2^{\frac{k}{2}})^2}{4\left\{\frac{\rho_1^h-1}{(\log|\rho_1|)^2}-\frac{h}{\log|\rho_1|}\right\}\left\{\frac{\rho_2^k-1}{(\log|\rho_2|)^2}-\frac{k}{\log|\rho_2|}\right\}(\log|\rho_1|)^2(\log|\rho_2|)^2}$$

for $|s|\geqslant h$ and $|t|\geqslant k$,

$$\frac{\rho_2^{|t|}(\rho_2^{-\frac{k}{2}}-\rho_2^{k/2})\left\{\frac{\rho_1^{|s|}(\rho_1^h-2)+\rho_1^{h-|s|}}{(\log|\rho_1|)^2}-\frac{2(h-|s|}{\log|\rho_1|)}\right\}}{4(\log|\rho_2|)^2\left\{\frac{\rho_1^h-1}{(\log|\rho_1|)^2}-\frac{h}{\log|\rho_1|}\right\}\left\{\frac{\rho_2^k-1}{(\log|\rho_2|)^2}-\frac{k}{\log|\rho_2|}\right\}} \quad \ldots(3·1·3)$$

for $|s|<h$ and $|t|\geqslant k$,

$$\frac{\rho_1^{|s|}(\rho_1^{-\frac{h}{2}}-\rho_1^{\frac{h}{2}})^2\left\{\frac{\rho_2^{|t|}(\rho_2^k-2)+\rho_2^{k-|t|}}{(\log|\rho_2|)^2}-\frac{2(k-|t|)}{\log|\rho_2|}\right\}}{4(\log|\rho_1|)^2\left\{\frac{\rho_1^h-1}{(\log|\rho_1|)^2}-\frac{h}{\log|\rho_1|}\right\}\left\{\frac{\rho_2^k-1}{(\log|\rho_2|)^2}-\frac{h}{\log|\rho_2|}\right\}}$$

for $|s|\geqslant h$ and $|t|<k$.

Proof:

When $\rho(s, t)=\rho_1^{|s|}\rho_2^{|t|}$ the numerator of (3·1·1) becomes

$$\left\{\int_{s-\frac{h}{2}}^{s+\frac{h}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho_1^{|x-u|}\,dxdu\right\}\left\{\int_{t-\frac{k}{2}}^{t+\frac{k}{2}}\int_{-\frac{k}{2}}^{\frac{k}{2}}\rho_2^{|y-v|}\,dyhv\right\}$$

...(3·1·4)

Employing (2·2·1) to each of the integrals in (3·1·4) and considering the four possible cases (*i*) $|s|<h$, $|t|<k$; (*ii*) $|s|\geqslant h$, $|t|\geqslant k$; (*iii*) $|s|<h$, $|t|\geqslant k$; (*iv*) $|s|\geqslant h$, $|t|<k$ the four results in (3·1·3) are easily obtained.

Remark :

It is easily noticed that $\rho_I(s, t)$ is continuous at the point (h, k) also.

Corollary 1 :

If $\rho(s, t)=\rho_1^{|s|}\rho_2^{|t|}$ and $h=k=1$, then

$$\rho_I(s, 0)=\begin{cases} a\rho_1^{|s|-1} \text{ for } |s|\geqslant 1, \text{ where } a=\dfrac{(\rho_1-1)^2}{2(\rho_1-1-\log|\rho_1|)} \\ 1 \text{ for } s=0 \end{cases} \quad ...(3·1·5)$$

and

$$\rho_I(0, t)=\begin{cases} b\rho_2^{|t|-1} \text{ for } |t|\geqslant 1, \text{ where } b=\dfrac{(\rho_2-1)^2}{2(\rho_2-1-\log|\rho_2|)} \\ 1 \text{ for } t=0 \end{cases} \quad ...(3·1·6)]$$

In particular

$$\rho_I(1, 0)=a\,; \quad \rho_I(0, 1)=b \quad ...(3·1·7)$$

Proof :

By putting $t=0$ in (3·1·3) with $|s|\geqslant 1$, and $s=0$ with $|t|\geqslant 1$, (3·1·5) and (3·1 6) follow. (3·1·7) follows by putting $s=1$ and $t=1$ in (3·1·5) and (3·1·6).

Corollary 2 :

If $\rho(s, t)=\rho_I^{|s|}\rho_2^{|t|}$, then

$$\rho_I(s, t)=\rho_I(s, 0)\,\rho_i(0, t) \quad ...(3·1·8)$$

Proof:

From (3·1·4) it is obvious that the numerator and denominator of (3·1·1) are products of two terms, each of which reduces to $\rho_I(s, 0)$ and $\rho_I(0, t)$.

3·2. Results on Estimation :

THEOREM 3 : When the auto-correlation $\rho(s, t)$ of the Basic process is $\rho_1^{|s|} \rho_2^{|t|}$, then consistent estimates $r_1 \equiv r(1, 0)$, $r_2 \equiv r(0, 1)$ of $\rho_1 \equiv \rho(1, 0)$ and $\rho_2 \equiv \rho(0, 1)$ are given by

$$R(1, 0) = \hat{\rho}_I(1, 0) = \hat{a} = \frac{(r_1 - 1)^2}{2(r_1 - 1 - \log |r_1|)} \qquad \ldots(3\cdot2\cdot1)$$

and

$$R(0, 1) = \hat{\rho}_I(0, 1) = \hat{b} = \frac{(r_2 - 1)^2}{2(r_2 - 1 - \log |r_2|)} \qquad \ldots(3\cdot2\cdot2)$$

Proof :

This follows from (3·1·7) and the fact that $r_1 = r(1, 0)$ and $r_2 = r(0, 1)$ are taken as the estimates of ρ_1 and ρ_2.

3·3. Results of Standard Errors :

Discrete Parameter Case :

THEOREM 4 : When the auto-correlation $\rho(s, t)$ of lag (s, t) of the Basic process is $\rho_1^{|s|} \rho_2^{|t|}$, then the variance of the sample auto-correlation of lag (s, t) of the Integrated process is

$$\text{Var}\,\{R(s, t)\} \sim \frac{1}{(m - |s|)(n - |t|)}$$

$$\Bigg\{ (1 + a^2 b^2 \rho_1^{2|s|-2} \rho_2^{2|t|-2}) \left(1 + \frac{2a^2}{1-\rho_1^2}\right)\left(1 + \frac{2b^2}{1-\rho_2^2}\right)$$

$$+ \rho_1^{2|s|-2} \rho_2^{2|t|-2} \left[2a\rho_1 + a^2\left(2|s| - 2 + \frac{1+\rho_1^2}{1-\rho_1^2}\right)\right]$$

$$\times \left[2b\rho_2 + b^2\left(2|t| - 2 + \frac{1+\rho_2^2}{1-\rho_2^2}\right)\right] - 4ab\rho_1^{2|s|-3} \rho_2^{2|t|-3}$$

$$\times \left[2a\rho_1 + a^2\left(|s| - 2 + \frac{1+\rho_1^2}{1-\rho_1^2}\right)\right]\left[2b\rho_2 + b^2\left(|t| - 2 + \frac{1+\rho_2^2}{1-\rho_2^2}\right)\right]\Bigg\}$$

$$\ldots(3\cdot3\cdot1)$$

Proof :

It is seen from formula (4·1·3) of [1] that

$$\text{Var}\{R(s,t)\} \sim \frac{1}{(m-|s|)(n-|t|)} \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} [1+2\rho_I^2(s,t)]\,\rho_I^2(u,v)$$

$$+\sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} \rho_I(u-s,v-t)\,\rho_I(u+s,v+t)$$

$$-4\rho_I(s,t)\sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} \rho_I(u,v)\,\rho_I(u-s,v-t) \qquad \ldots(3\cdot3\cdot2)$$

Employing (3·1·8), (3·1·5), (3·1·6), (2·2·5) and (2·2·6) of Lemma 2 and noting that λ of (2·2·4) is either a/ρ_1 or b/ρ_2, it is easily seen that after simplification (3·3·1) follows.

Corollary 3 :

If $\rho(s,t)=\rho_1^{|s|}\ \rho_2^{|t|}$, then

$$\text{Var}\{R(s,0)\} \sim \frac{1}{(m-|s|)n}$$

$$\left\{(1+2a^2\rho_1^{2|s|-2})\left(1+\frac{2a^2}{1-\rho_1^2}\right)\left(1+\frac{2b^2}{1-\rho_2^2}\right)\right.$$

$$+\rho_1^{2|s|-2}\left[2a\rho_1+a^2\left(2|s|-2+\frac{1+\rho_1^2}{1-\rho_1^2}\right)\right]\left(1+\frac{2b^2}{1-\rho_2^2}\right)$$

$$\left.-4a\rho_1^{2|s|-2}\left[2a\rho_1+a^2\left(|s|-2+\frac{1+\rho_1^2}{1-\rho_1^2}\right)\right]\left(1+\frac{2b^2}{1-\rho_2^2}\right)\right\} \qquad \ldots(3\cdot3\cdot3)$$

and

$$\text{Var}\{R(0,t)\} \sim \frac{1}{m(n-|t|)}$$

$$\left\{(1+2b^2\rho_2^{2|t|-2})\left(1+\frac{2a^2}{1-\rho_1^2}\right)\left(1+\frac{2b^2}{1-\rho_2^2}\right)\right.$$

$$+\rho_2^{2|t|-2}\left[2b\rho_2+b^2\left(2|t|-2+\frac{1+\rho_2^2}{1-\rho_2^2}\right)\right]\left(1+\frac{2a^2}{1-\rho_1^2}\right)$$

$$\left.-4b\rho_2^{2|t|-2}\left[2b\rho_2+b^2\left((|t|-2+\frac{1+\rho_2^2}{1-\rho_2^2}\right)\right]\left(1+\frac{2a^2}{1-\rho_1^2}\right)\right\} \qquad \ldots(3\ 3\cdot4)$$

In particular

$$\text{Var}\,\{R(1,0)\} \sim \frac{1}{(m-1)n}\left\{(1+2a^2)\left(1+\frac{2a^2}{1-\rho_1^2}\right)\right.$$
$$\left.+\left(2a\rho_1+a^2\frac{1+\rho_1^2}{1-\rho_1^2}\right)-8a\left(a\rho_1+\frac{a^2\rho_1^2}{1-\rho_2^2}\right)\right\}\left(1+\frac{2b_2}{1-\rho_2^2}\right) \quad \ldots(3\cdot3\cdot5)$$

and

$$\text{Var}\,\{R(0,\ 1)\} \sim \frac{1}{m(n-1)}\left\{(1+2b^2)\left(1+\frac{2b^2}{1-\rho_2^2}\right)\right.$$
$$\left.+\left(2b\rho_2+b^2\frac{1+\rho_2^2}{1-\rho_2^2}\right)-8b\left(b\rho_2+\frac{b^2\rho_2^2}{1-\rho_2^2}\right)\right\}\left(1+\frac{2a^2}{1-\rho_1^2}\right) \quad \ldots(3\cdot3\cdot6)$$

Proof :

Writing expressions similar to (3·3·2) with $s=0$ or $t=0$ and employing (2·2·5) and (2·2·6) it is seen that (3·3·3) and (3·3·4) follow. Further by putting $s=1$ and $t=1$ in (3.3.3) and (3·3·4) the results in (3·3·5) and (3·3·6) are obtained.

Continuous Parameter Case :

THEOREM 5 : If the auto-correlation $\rho(s, t)$ of lag $(s,\ t)$ of the Basic process is $\rho_1^{\,|s|}\,\rho_2^{\,|t|}$, the variances of the sample auto-correlations of lags(1,0) and (0, 1) are

$$\text{Var}\,\{R(1,\ 0)\} \sim \left\{\frac{\partial\psi(\rho_1)}{\partial\rho_1}\right\}^2 \text{Var}\,\{r(1,\ 0)\} \quad \ldots(3\cdot3\cdot7)$$

and

$$\text{Var}\,\{R(0,\ 1)\} \sim \left\{\frac{\partial\psi(\rho_2)}{\partial\rho_2}\right\}^2 \text{Var}\,\{r(0,\ 1) \quad \ldots(3\cdot3\cdot8)$$

where

$$\psi(\rho)=\frac{(\rho-1)^2}{2(\rho-1-\log|\rho|)}.$$

Proof :

Writing $\psi(r) = \dfrac{(r-1)^2}{2(r-1-\log|r|)}$ and noting (3·2·1) and (3·2·2) that $\delta R \doteq \dfrac{\partial\psi(r)}{\partial r}\,\delta r$, (3·3·7) and (3·3·8) are obtained.

For the variance of $r(1,\ 0)$ and $r(0,\ 1)$ results from [1] are used for observations at discrete points of time c.f. formulae (5·1·2) to (5.1.4) of [1] and for observations at continuous points of time (c.f. fomulae (5·2·2) to (5·2·4) of [1]).

4. Application to Mercer and Hall wheat data

Mercer and Hall wheat data relates to observations on the yield of wheat on small rectangular plots each 11 ft. by 10·82 ft. arranged in a rectangular block of 20 rows and 25 columns making a total of 500 observations. This data was analysed and the auto-correlations for different lags are given in Whittle [6, p. 443]. R(1, 0) and R(0, 1) of the data which are 0.5252 and 0.2923 are taken by him as estimates of ρ_1 and ρ_2 in the theoretical auto-correlation function $\rho_1^{|s|}\ \rho_2^{|t|}$ he assumed. By removing a trend from West to East represented approximately by

$$Y = 3{\cdot}9485 - 0{\cdot}019041\ (X-13)$$

where Y represents the expected mean yield in each column and X is an auxiliary variable taking values 1 to 25 corresponding to the columns from West to East, Patankar obtained a modified data. He then estimated the auto-correlation function $\rho_1^{|s|}\ \rho_2^{|t|}$ by taking R(1, 0) and R(0, 1) of this modified data which are 0·3094 and 0·1557 as the estimates of ρ_1 and ρ_2. He stated that the estimates thus found gave a satisfactory fit to the modified data.

4·1. Estimate of the Auto-correlation Function :

By applying the results of THEOREM 3, the auto-correlation function is estimated as follows :

$$0{\cdot}5252 = R(1,\ 0) = \hat{\rho}_I(1,\ 0) = \frac{(\hat{\rho}_1 - 1)^2}{2(\hat{\rho}_1 - 1 - \log|\hat{\rho}_1|)} \quad \text{and}$$

$$0{\cdot}2923 = R(0,\ 1) = \hat{\rho}_I\ (0,\ 1) = \frac{(\hat{\rho}_2 - 1)^2}{2(\hat{\rho}_2 - 1 - \log|\hat{\rho}_2|)}.$$ Using Fisher and Yates tables [3] for $-\log|\rho|$ and solving the equations by numerical methods, we have $\hat{\rho}_1 = 0{\cdot}3462$ and $\hat{\rho}_2 = 0{\cdot}1030$. The theoretical auto-correlation function is thus found to be $(0{\cdot}3462)^{|s|}\ (0{\cdot}1030)^{|t|}$.

4·2. Standard Errors :

Standard errors of the estimates are calculated from the formulae (3·3·5) and (3·3·6) in the discrete case and from the formulae (3 3·7) and (3·3·8) in the continuous case, employing the results in [1] (c.f. (5·2·2) to (5·2·4) for Var $r(1,\ 0)$ and $r(0,\ 1)$, in [1]).

Taking the effect of integration into account the standard errors of R(1, 0) and R(0, 1) as calculated from (3·3·5) and (3·3·6) by treating the parameter as discrete are 0·0752 and 0·0704, whereas they are 0·0193 and 0·0343 when calculated from formulae (3·3·7) and (3·3·8) treating the parameters as continuous.

The estimates found in section 4 1 and the standard errors of the estimates calculated are given in the following table along side with the estimates of Whittle and Patankar obtained by not taking the effect of integration into account.

5. Conclusion :

The method adopted by Patankar is to estimate ρ_1 and ρ_2 from the modified data obtained by eliminating a trend which he assumed in the original data. The method adopted in this paper is to estimate ρ_1 and ρ_2 from the original data itself, but by taking into account what is called "the effect of integration".

From the table it is clear that the estimates obtained from the two approaches are close. Further the standard errors are also reasonably small and are less in the continuous parameter case than in the discrete parameter case. The estimates no doubt get improved by taking into account the effect of integration but the standard errors increase naturally.

It is possible that a superimposed trend also may accompany an "integrated process". But this is not attempted here.

Thus whenever observations relate to integrated values over a small area, it seems to be meaningful to take the effect of such integration into account in estimating, from data which represent integrated values over small neighbourhoods of points and by treating the Basic process as a continuous parameter process.

I thank the referee for some of his helpful comments.

Summary

In this paper the effect of taking *integrated* observations over a small neighbourhood of a point instead of *point* observations is studied from the point of view of empirical analysis of data. It is found that 'de-integration' of the data in general leads to more effective results.

The results of the problem studied in the general case are applied to the Mercer and Hall wheat data which was previously used by Whittle and Patankar for illustration. The standard errors of the estimates are obtained and a comparison is made.

Table showing the different Estimates of ρ_1 and ρ_2 and their Standard Errors

(The theoretical model assumed for the auto-correlation function is $\rho_1^{|s|} \rho_2^{|t|}$)

	Estimates		Standard Errors				Remarks
			Taking the effect of integration into account S.E. of		*Without taking the effect of integration into consideration S.E. of*		
	$\hat{\rho}_1$	$\hat{\rho}_2$	$\hat{\rho}_1$	$\hat{\rho}_2$	$\hat{\rho}_1$	$\hat{\rho}_2$	
Avadhani	0·3462	0·1030	0·0193	0·0343			Treating the parameters as continuous.
			0·0752	0·0704			Treating the parameters as discrete.
			0·0359	0·0616			Taking the parameters as continuous but using the discrete parameter approximation for Var (r_1) and Var (r_2).
					0·0433	0·0514	Treating the parameters as discrete.
					0·0288	0·0288	Continuous parameter
					0·0510	0·0510	Discrete parameter.
Patankar	0·3094	0·1557					
Whittle	0·5252	0·2923					

REFERENCES

1. Avadhani, T. V. (1968) : Sampling Errors of Covariances and Correlations for K-dimensional Stationary Stochastic Processes (to be published in Sankhya);

2. Bartlett, M. S. (1955) : Stochastic Processes : Methods and Applications (Cambridge);

3. Fisher, R. A. and Yates, F. (1953) : Statistical Tables for Biological, Agricultural and Medical Research (Oliver and Boyd);

4. Mercer, W.B. and Hall, A.D. (1911) : The Experimental Error of Field Trials (Jour. Agr. Sci. 4, pp. 107-32);

5. Patankar, V. N. (1954) : The Goodness of Fit of Frequency Distributions obtained from Stochastic Processes (Biometrika, Vol. 41, Parts 3 and 4, pp. 450-62);

6. Whittle, P. (1954) : On Stationary Processes in the Plane (Biometrika, Vol. 41, Parts 3 and 4, pp. 434-49).